Automorphic Forms

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Our references are to the relevant definitions in [Int].
Let G be a reductive group over a global field F.

Definition (6.5)**.** *If F is a number field then a function*

$$\phi : G(\mathbb{A}_F) \to \mathbb{C}$$

is an automorphic form on G if it is

- *Smooth*
- *Moderate growth*
- $G(F)$ *left invariant*
- K*-finite*
- $Z(\mathfrak{g})$*-finite*

Definition (6.7)**.** *If F is a function field then*

$$\phi : G(\mathbb{A}_F) \to \mathbb{C}$$

is an automorphic form on G if it is

- $G(F)$ *left invariant*
- *Invariant on the right under the action of some compact open subgroup of* $G(\mathbb{A}_F)$
- *The* $\mathbb{C}$ *span of*
$$x \mapsto \phi(xg) : g \in G(\mathbb{A}_F)$$
is an admissable representation

So thats a lot of words that we should define now

1 Smooth

1.1 Archimedean Local Field

Let H be an AAG over an archimedean local field k. It is a fact that every archimedean local field is either $\mathbb{R}$ or $\mathbb{C}$. H is AAG and hence linear so we can embed it in GL_n, thus a smooth function from $H(k) \to \mathbb{C}$ is a smooth function in the ordinary differential topology sense from the manifold $GL_n(\mathbb{R})$ or $GL_n(\mathbb{C})$.

proof

1.2 Non-Archimedean Local Field

If H is an affine algebraic group (AAG) over a non-archimedean local field k then $H(k)$ is totally disconnected and we say that

$$f : H(k) \to \mathbb{C}$$

is smooth if it is locally constant in the induced topology on $H(k)$ from the topology on k.

explain what that topology is

1.3 Global Field

Let H be an AAG over a global field k and ν a place

Theorem.

$$H(k_\nu) \cong H_{k_\nu}(k_\nu)$$

Therefore a function $f : H(k_\nu) \to \mathbb{C}$ is smooth if it is smooth as a function $f : H_{k_\nu}(k_\nu) \to \mathbb{C}$ as defined for archimedean and non-archimedean local fields above.

Chenyan assures me all the connonical isos work out but I could also try and prove that base changing reductive is reductive etc

1.4 Adelic Smoothness

Recalling that G is a reductive group over a global field F we make the following definitions:

For the non-archimedean places we define

$$C^\infty(\mathbb{A}_F^\infty) \quad := \quad \bigotimes_{\nu \nmid \infty}{}' C^\infty(G(F_\nu))$$

And for the archimedean places we define

$$C^\infty(G(F_\infty)) \quad := \quad C^\infty\left(\prod_{\nu \mid \infty} G(F_\nu)\right)$$

For the full Adele we define

$$C^\infty(\mathbb{A}_F) \quad := \quad C^\infty(G(F_\infty)) \otimes C^\infty(G(\mathbb{A}_F^\infty))$$

A function is ***smooth*** if it is in one of these sets for the appropriate domain. Note that this gives functions with codomain being the tensor product of a bunch of $\mathbb{C}$'s over $\mathbb{C}$ which is isomorphic to $\mathbb{C}$, so we are justified in making this identification.

Another remark is that in our notation ∞ simply stands for the collection of archimedean places.

2 The Rest

Invariance: A function

$$\phi : G(\mathbb{A}_F) \to \mathbb{C}$$

is (left) ***invariant*** under the action of a subgroup $H \leq G(\mathbb{A}_F)$ when $\forall \gamma \in H$ we have that

$$\phi(\gamma g) = g \quad \forall g \in G(\mathbb{A}_F)$$

For the above definitions we view $G(F) \leq G(\mathbb{A}_F)$ via the diagonal map.

2.1 Adelic Number Field

- **Moderate growth:** First we define a norm on $G(\mathbb{A}_F)$. Becuase G is reductive it is in particular linear, we therefore fix a closed embedding $\iota' : G \to GL_n$, which gives another closed embedding $\iota : G \to SL_{2n}$ by

  $$g \mapsto \begin{pmatrix} \iota'(g) & \\ & \iota'(g^{-1})^t \end{pmatrix}$$

  and the norm is

  $$\|g\| = \prod_{\nu} \sup_{1 \leq i,j \leq 2n} |\iota(g)_{ij}|_\nu$$

  There is an abuse of notation here $\iota(g)_{ij}$ should actually be the projection onto the ν place and then take the norm. Note that we have made some choices of embeddings here however the class of functions that is of moderate growth is actually independent of the embedding. Then a function $f : G(\mathbb{A}_F) \to \mathbb{C}$ is of ***moderate growth*** if there exists some $c, r \in \mathbb{R}_{>0}$ such that for every $g \in G(\mathbb{A}_F)$

  $$|f(g)| \leq c\|g\|^r$$

- **K-finite:** We choose two subgroups this time; $K_\infty \leq G(F_\infty), K^\infty \leq G(\mathbb{A}_F^\infty)$ where as before K_∞ is a maximal compact subgroup, and K^∞ is some compact open subgroup. We then define $K = K_\infty K^\infty$ the direct product. We then say that a function $f : G(\mathbb{A}_F) \to \mathbb{C}$ is ***K-finite*** if

  $$\dim[span_{\mathbb{C}}\{x \mapsto f(xk) : k \in K\}] < \infty$$

  I have been assured that this is infact independent of the choice made.

- **$Z(\mathfrak{g})$-finite:** $Z(\mathfrak{g})$ is the center of the Lie algebra associated to $G(F_\infty)$ and we say that a vector $f \in V$ is $Z(\mathfrak{g})$***-finite*** if $Z(\mathfrak{g})f$ is finite dimensional.

  am I right

  again whats our representation here, V and the action

2.2 Adelic Function Field

- **The $\mathbb{C}$ span of**

  $$\{x \mapsto \phi(xg) : g \in G(\mathbb{A}_F)\}$$

  is an admissable representation: Recall that an ***admissable representation*** of a topological group (actual group) H is a representation (π, V) such that for every $v \in V$ the stabilizer $\text{stab}_H(v)$ is open in G and for every open subgroup $K \subseteq H$ $dimV^K < \infty$.

  of what, the adelic

Remark. In the archimedain subcase [Int] gives explicitly that the functions are invariant under some arithmentic subgroup. The general definition of automorphic form does not have this restriction. Moreover the choice of K does not effect the collection of automorphic forms. The correct analogie is that if we required the functinos to be K_∞ invariant functions. Then we recover the more familar notion, in particular modular forms etc.

3 Modular and Maas Forms

One might ask if there is a special case in which these automorphic forms yeild modular forms. In fact no, the space of automorphic forms is larger than just modular forms, however it gives the space of Maas forms (or modular and Maas forms, depending on convention). We follow [Bum97][3.2] for the exposition here.

We first let $G = GL_2(\mathbb{R})^+$ the positive determinant matricies. Let Γ be a discrete subgroup of $SL_2(\mathbb{R})$ containing the negative identity matrix and such that $\Gamma \setminus \mathcal{H}$ has finite volume ($\mathcal{H}$ is the upper half of the complex plane and Γ acts via linear fractionanl transformations as usual). Let $Z(\mathbb{R})$ be the center of G consisting of scalar matricies and let $K = SO(2)$ be the maximal compact subgroup.

are reductive groups always Lie groups in the infinite places? Is the atlas SMOOTH

Definition. *A function*

$$\phi : \mathcal{H} \to \mathbb{C}$$

that is holomorphic, and satisfies

$$\phi(\gamma.z) = \chi(\gamma)(cz+d)^k \phi(z), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

is called a modular form for Γ with character χ of weight k.

Definition. *Define a right action of G on functions $\mathcal{H} \to \mathbb{C}$ via*

ok now im confused why this would even a priori define an action becuase you might get infinities..?

$$(f|_k g)(z) \quad := \quad \left(\frac{c\bar{z}+d}{|cz+d|} \right)^k f\left(\frac{az+b}{cz+d} \right)$$

and define the weight k laplacian

$$\Delta_k : C^\infty(\mathcal{H}) \to C^\infty(\mathcal{H})$$

$$\Delta_k = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \frac{\partial}{\partial x}$$

A Maas form of weight k is a smooth function $f : \mathcal{H} \to \mathbb{C}$ such that

- *$f|_k g = \chi(g) f$ for all $g \in G$*
- *f is an Eigenform of Δ_k (with eigenvalue λ)*
- *f has moderate growth at the cusps of Γ*

Lemma. *All modular forms are Maas forms.*

We remark that the two are sometimes defined to be complimentary. So it is sufficient to attach an automorphic form to any Maas form. Define

$$F(g) = (f|_k g)(i) : G \to \mathbb{C}$$

To bring this a little bit closer to the language that we set things up in above let us elaborate further.

First we are considering here only the infinite place of the group scheme GL_2^+ over $\mathbb{Q}$ which happens to be $\mathbb{Q}_\infty = \mathbb{R}$. This is valid (and indeep how [Int] initially sets things up), because we can define the automorphic form on a basis, and make it trivial on all the other places and it thereby trivially satisfies the smooth, moderate growth, invariance and finiteness conditions at those places (for the finiteness of $Z(\mathfrak{g})$ see [Bum97] or take it as a definition that an "automorphic form" can be defined similarly for only the infinite place).

does the plus mess with anything representability wise...

We should elaborate on this much more however, by translating and checking each of the conditions.

Smooth. The Maas form f is smooth as a function on the upper half plane, therefore F is smooth on G becuase it is the multiplication of two smooth functions.

I dont feel too comfortable with this but mainly becuase I dont know that the character is smooth or that this thing is really an action (above)

Z-Finite. f is an eigenfunction of Δ_k hence F is an eigenfunction of Δ (Chpt 2, eqn 1.30) the center of the universal enveloping algebra is the polynomial algebra in Δ.

K-Finite.

Lemma. *If* $a = \begin{pmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{pmatrix} \in SO(2)$ *then*

$$F(ga) = e^{2\pi ik} F(g)$$

It follows that for all $k \in K$ we have $\rho(k)F \propto F$ and so $\dim span_{k\in SO(2)}\{\rho(k)F\} = 1$. Hence F is K finite.

3.1 Summary

- Smooth: Smooth
- Moderate growth: Moderate growth
- $G(F)$ left invariant: A minimal amount of invariance
- K-finite: Invarience under some subgroup
- $Z(\mathfrak{g})$-finite: Satisfies some differential equation

References

[Bum97] Daniel Bump. *Automorphic Forms and Representations.* Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1997.

[Int] *An Introduction to Automorphic Representations.*